Damascus University
Higher Institute of Earthquake Studies and Researches

Advanced Math-Modeling and Numerical Analysis

Lec.06

Plane Stress and Plane Strain Stiffness Equations
Constant Strain Triangle (CST)

Dr. Reem Alsehnawi

reemsalman_seh@hotmail.com

We- 18/3/2020

Introduction:

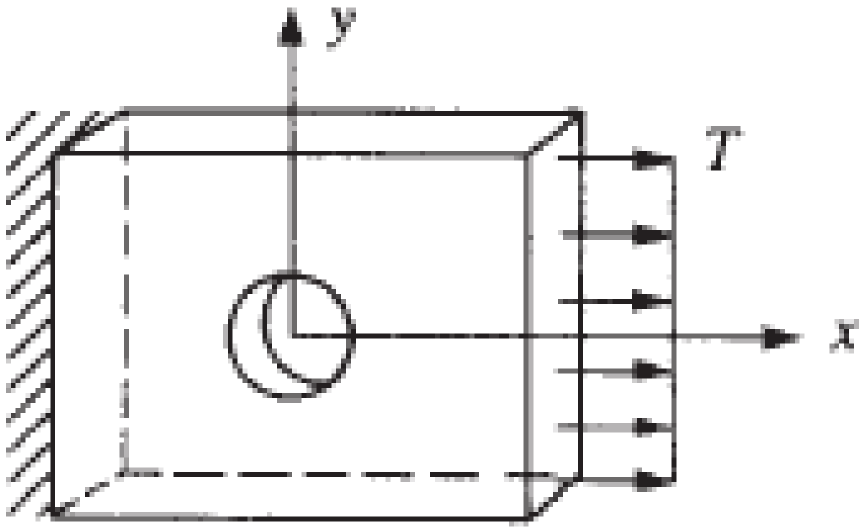
Two-dimensional (planar) elements are defined by three or more nodes in a two-dimensional plane (that is, x-y).

We begin with the development of the stiffness matrix for a basic two-dimensional or plane finite element, called the constant-strain triangular element CST.

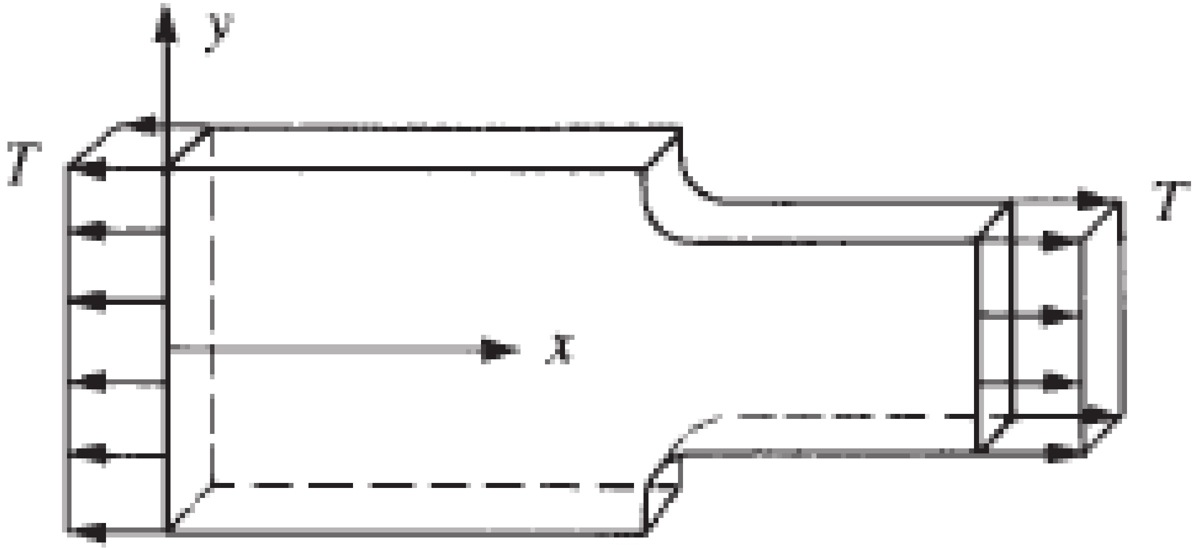
Basic concept of plan stress and plan strain:

Plane stress: a state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero.

members that are thin (those with a small z dimension compared to the in-plane x and y dimensions) and whose loads act only in the x-y plane can be considered to be under plane stress.



(a)

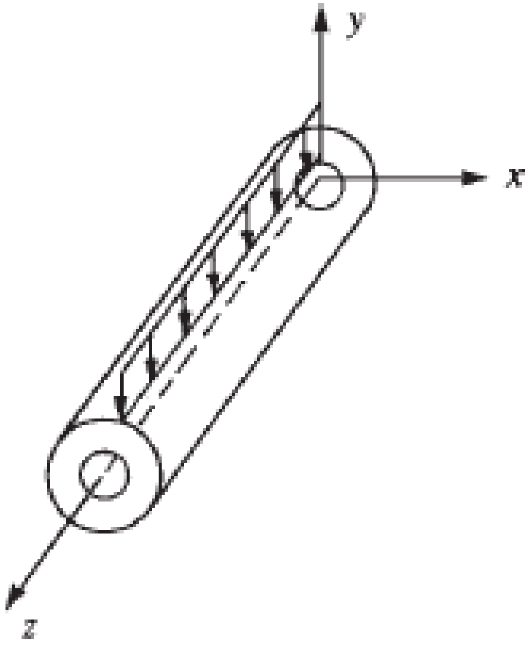


(b)

Basic concept of plan stress and plan strain:

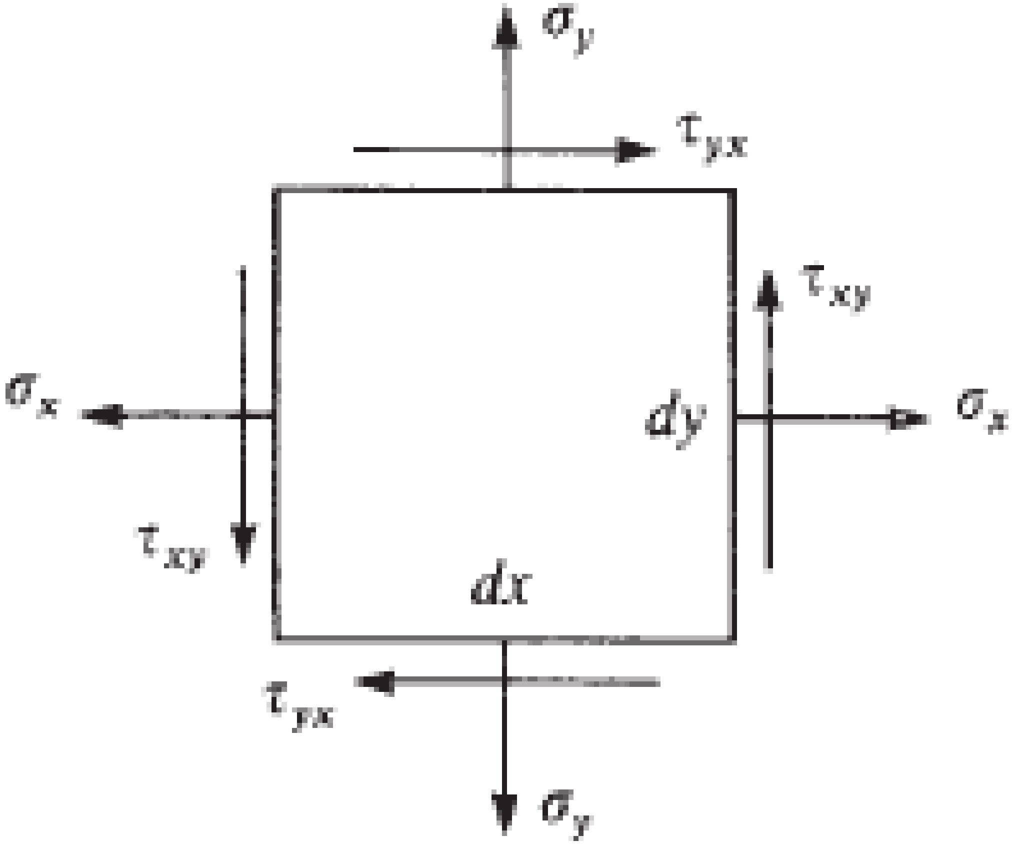
Plane strain: a state of strain in which the strain normal to the x-y plane ε_z and the shear strains γ_{xz} and γ_{yz} are assumed to be zero.

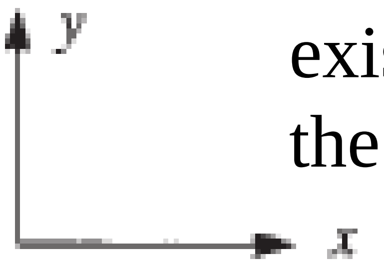
Strain are realistic for long bodies (say, in the z direction) with constant cross-sectional area subjected to loads that act only in the x and/or y directions and do not vary in the z direction.



Two-Dimensional State of Stress and Strain

The essential concepts of two-dimensional stress and strain





Three independent stresses exist and are represented by the vector column matrix

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

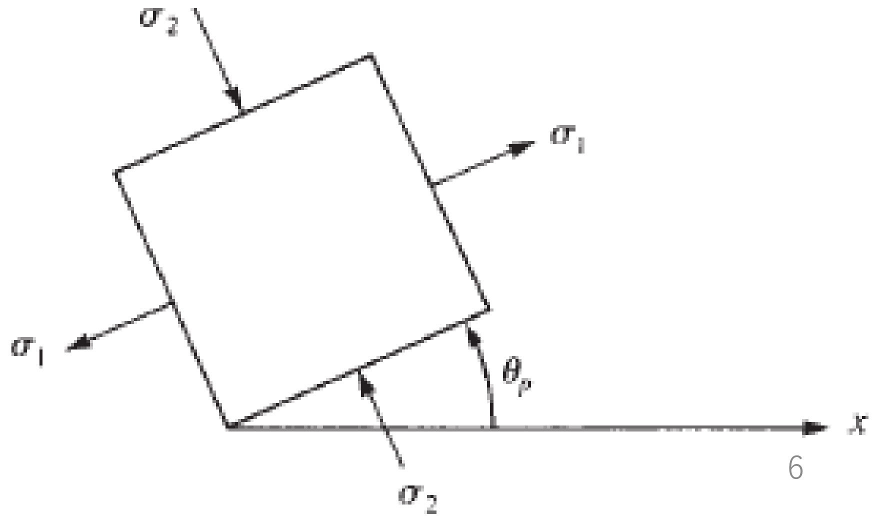
The principal stresses:
(maximum and minimum normal stresses in the two-dimensional plane):

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{max}$$

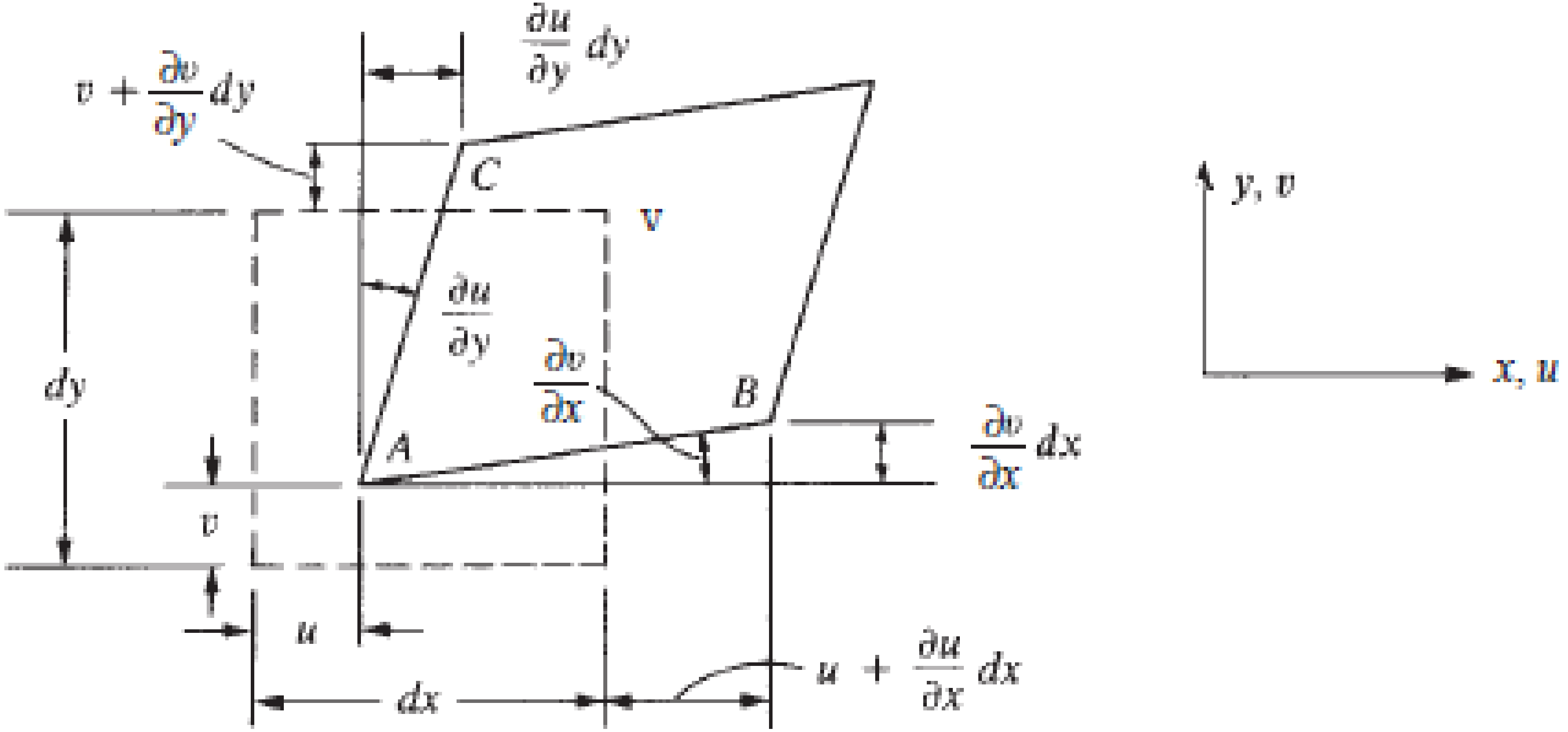
$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{min}$$

The shear stress is zero on the planes having principal normal stresses.

The principal angle θ_p: $$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



general two-dimensional state of strain:



The general two-dimensional state of strain at some point in a structure.
The element displaced by amounts u and v in the x and y directions at point A, and extend an incremental amount $(\partial u/\partial x)$ dx along line AB.
$(\partial \vartheta/\partial y)$dy along line AC in the x and y directions, respectively.
point B moves upward an amount $(\partial \vartheta/\partial y)$dx with respect to A.
point C moves to the right an amount $(\partial u/\partial x)$dywith respect to A.

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

The strain γ_{xy} is the change in the original right angle made between *dx* and *dy* when the element undergoes deformation. The strain γ_{xy} is then called a shear strain.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

For plane stress, $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

We first consider the change in length of the element in the *x* direction due to the independent stresses σx, σy, and σz.

We assume that the resultant strain in a system due to several forces is the algebraic sum of their individual effects.

The stress in the *x* direction produces a positive strain

Hooke's law

$$\sigma = E\varepsilon,$$

$$\varepsilon_x' = \frac{\sigma_x}{E}$$

The positive stress in the *y* direction produces a negative strain in the *x* direction as a result of Poisson's effect given by

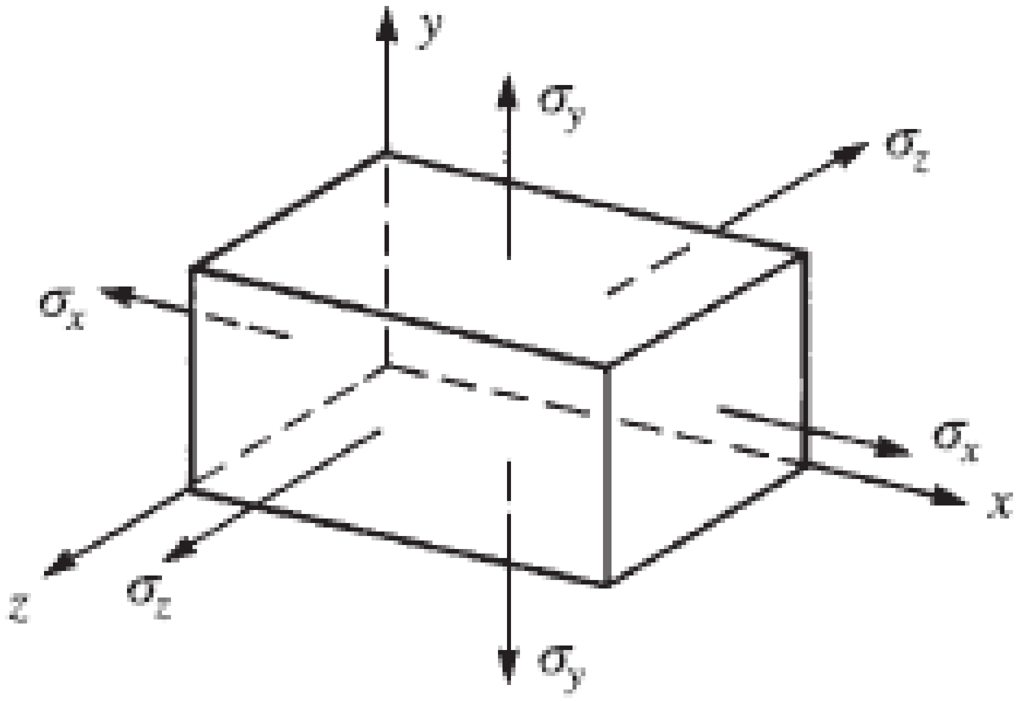
$$\varepsilon_x'' = -\frac{\nu\sigma_y}{E}$$

$$\varepsilon_x''' = -\frac{\nu\sigma_z}{E}$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$



Solving the equations for the normal stresses, we obtain:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[\varepsilon_x(1-\nu) + \nu\varepsilon_y + \nu\varepsilon_z\right]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_x + (1-\nu)\varepsilon_y + \nu\varepsilon_z\right]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_x + \nu\varepsilon_y + (1-\nu)\varepsilon_z\right]$$

For shear stress and strain; that is,

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx}$$

and $$G = \frac{E}{2(1+\nu)}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \times \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & & & & \frac{1-2\nu}{2} & 0 \\ \text{Symmetry} & & & & & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Stress-strain matrix

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & & & & \frac{1-2\nu}{2} & 0 \\ \text{Symmetry} & & & & & \frac{1-2\nu}{2} \end{bmatrix}$$

Stress–strain relationships for both plane stress and plane strain.

For plane stress,

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

Applying the previous equation to the three-dimensional stress–strain relationship, the shear strains $\gamma_{xz} = \gamma_{yz} = 0$, but $\varepsilon_z \neq 0$.

$$\{\sigma\} = [D]\{\varepsilon\}$$

where

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Stress–strain relationships for both plane stress and plane strain.

For plane strain,

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

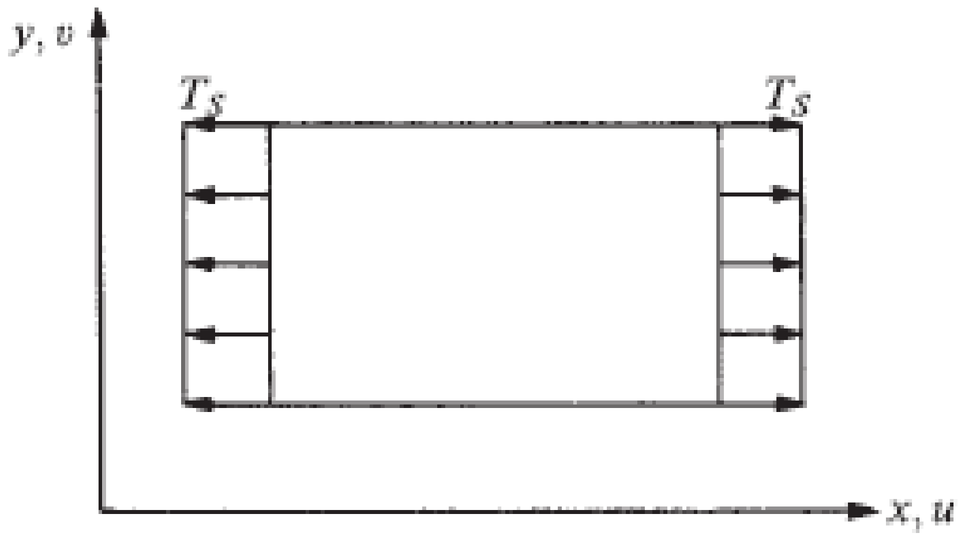
Applying the equation to the three-dimensional stress–strain relationship, the shear stresses $\tau_{xz} = \tau_{yz} = 0$, but $\sigma_z \neq 0$.

The stress–strain matrix:

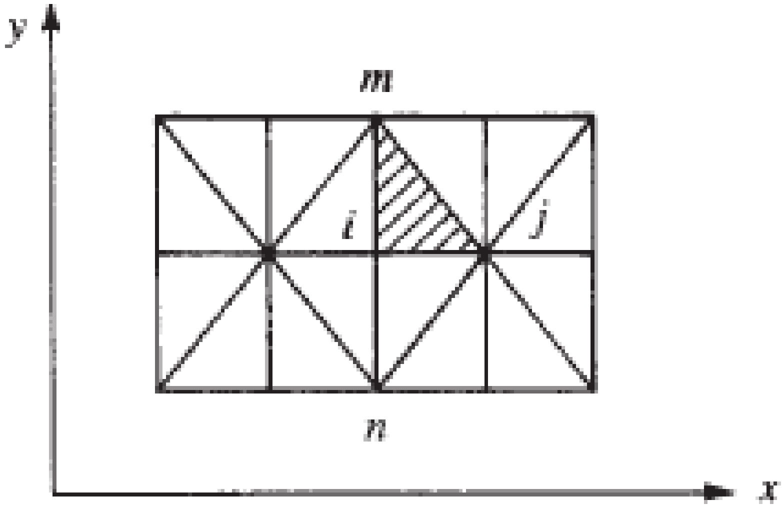
$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Constant Strain Triangle (CST or T3)

Stiffness Matrix and Equations:



Thin plate in tension

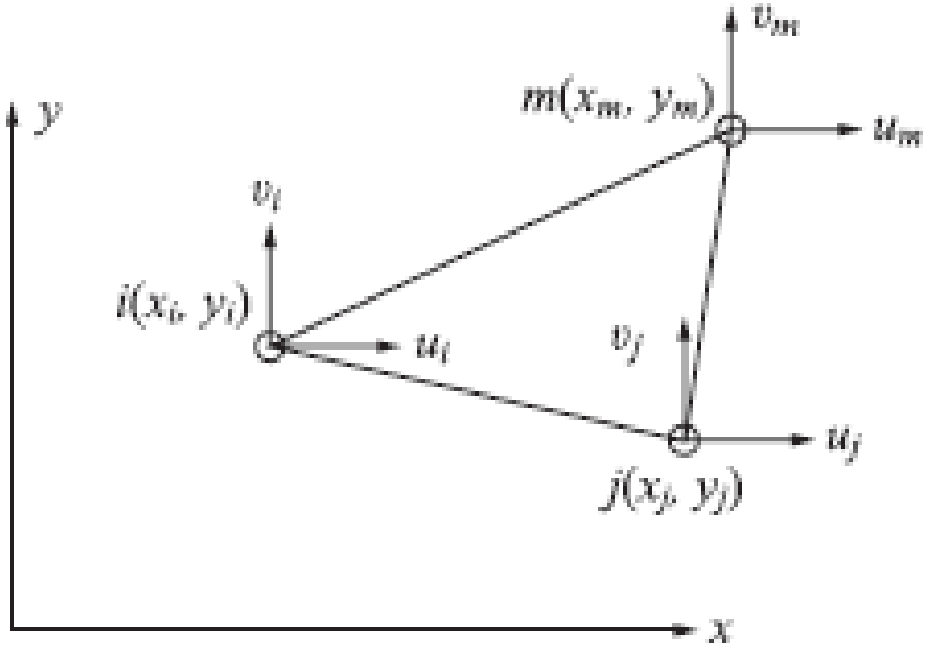


Discretized plate using triangular elements

Constant Strain Triangle (CST or T3)

Select Element Type:

$$\{d\} = \begin{Bmatrix} \{d_i\} \\ \{d_j\} \\ \{d_m\} \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$



Basic triangular element showing degrees of freedom

- Three nodes at the vertices of the triangle,
- numbered around the element in the counterclockwise direction.
- Each node has two degrees of freedom.
- Displacements *u* and *v* are assumed to be linear functions within the element.

Constant Strain Triangle (CST or T3)

Select Displacement Functions:

Selecting a linear displacement function for each element as,

$$u(x,y) = a_1 + a_2 x + a_3 y$$

$$v(x,y) = a_4 + a_5 x + a_6 y$$

General displacement function:

$$\{\psi\} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

To obtain the a’s; by substituting the coordinates of the nodal points into:

$$u_i = u(x_i, y_i) = a_1 + a_2 x_i + a_3 y_i$$

$$u_j = u(x_j, y_j) = a_1 + a_2 x_j + a_3 y_j$$

$$u_m = u(x_m, y_m) = a_1 + a_2 x_m + a_3 y_m$$

$$v_i = v(x_i, y_i) = a_4 + a_5 x_i + a_6 y_i$$

$$v_j = v(x_j, y_j) = a_4 + a_5 x_j + a_6 y_j$$

$$v_m = v(x_m, y_m) = a_4 + a_5 x_m + a_6 y_m$$

We can solve for the a’s beginning with the first three equations:

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \Rightarrow \{a\} = [x]^{-1}\{u\}$$

$$[x]^{-1} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \quad \text{where} \quad 2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

is the determinant of $[x]$, which on evaluation is

$$2A = x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j)$$

Here A is the area of the triangle, and

$$\alpha_i = x_j y_m - y_j x_m \qquad \alpha_j = y_i x_m - x_i y_m \qquad \alpha_m = x_i y_j - y_i x_j$$

$$\beta_i = y_j - y_m \qquad \beta_j = y_m - y_i \qquad \beta_m = y_i - y_j$$

$$\gamma_i = x_m - x_j \qquad \gamma_j = x_i - x_m \qquad \gamma_m = x_j - x_i$$

Having determined $[x]^{-1}$, we can now express

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

Similarly, using the last three equations, we can obtain

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix}$$

We will derive the general x displacement function u(x,y) of $\{\psi\}$

in terms of the coordinate variables x and y, known coordinate variables $\alpha_i, \alpha_j, \ldots, \gamma_m$, and unknown nodal displacements u_i, u_j, and u_m.

$$u(x,y) = a_1 + a_2 x + a_3 y \quad \Rightarrow \quad \{u\} = [1 \quad x \quad y]\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

$$\{u\} = \frac{1}{2A}[1 \quad x \quad y]\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

$$\{u\} = \frac{1}{2A}[1 \quad x \quad y]\begin{Bmatrix} \alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m \end{Bmatrix}$$

$$u(x,y) = \frac{1}{2A}\{(\alpha_i + \beta_i x + \gamma_i y)u_i + (\alpha_j + \beta_j x + \gamma_j y)u_j + (\alpha_m + \beta_m x + \gamma_m y)u_m\}$$

Similarly, replacing u_i by v_i, u_j by v_j, and u_m by v_m

$$v(x,y) = \frac{1}{2A}\{(\alpha_i + \beta_i x + \gamma_i y)v_i + (\alpha_j + \beta_j x + \gamma_j y)v_j + (\alpha_m + \beta_m x + \gamma_m y)v_m\}$$

To express these equations simpler for u and v in simpler form, we define

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

$$N_j = \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y)$$

$$N_m = \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)$$

$$u(x, y) = N_i u_i + N_j u_j + N_m u_m$$

$$v(x, y) = N_i v_i + N_j v_j + N_m v_m$$

$$\{\psi\} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

General displacements as functions of {d}, in terms of the shape functions N_i, N_j, and N_m.

$$\{\psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

Constant Strain Triangle (CST or T3)

Define the Strain/Displacement and Stress/Strain Relationships

Expressing the element strains and stresses in terms of the unknown nodal displacements.

Element Strains

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

Substituting displacement functions for u and v from

$$u(x, y) = a_1 + a_2 x + a_3 y$$

$$v(x, y) = a_4 + a_5 x + a_6 y$$

$$\varepsilon_x = a_2 \qquad \varepsilon_y = a_6 \qquad \gamma_{xy} = a_3 + a_5$$

The strains in the element are constant.

The element is then called a constant-strain triangle (CST).

Constant Strain Triangle (CST or T3)

Define the Strain/Displacement and Stress/Strain Relationships

$$\frac{\partial u}{\partial x} = \frac{1}{2A}(\beta_i u_i + \beta_j u_j + \beta_m u_m)$$

$$\frac{\partial v}{\partial y} = \frac{1}{2A}(\gamma_i v_i + \gamma_j v_j + \gamma_m v_m)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}(\gamma_i u_i + \beta_i v_i + \gamma_j u_j + \beta_j v_j + \gamma_m u_m + \beta_m v_m)$$

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \qquad [B_j] = \frac{1}{2A}\begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix} \qquad [B_m] = \frac{1}{2A}\begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix}$$

Constant Strain Triangle (CST or T3)

Define the Strain/Displacement and Stress/Strain Relationships

$$\{\varepsilon\} = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

Constant Strain Triangle (CST or T3)

Define the Strain/Displacement and Stress/Strain Relationships

Stress-Strain Relationship:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

For plane stress

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

$$\{\sigma\} = [D][B]\{d\}$$

For plane strain

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

where the stresses $\{\sigma\}$ are also constant everywhere within the element.

Element stiffness matrix for the CST element,

Consider the strain energy stored in an element,

$$U = \frac{1}{2}\int_V \sigma^T \varepsilon \, dV = \frac{1}{2}\int_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy}) dV$$

$$= \frac{1}{2}\int_V (\mathbf{E}\varepsilon)^T \varepsilon \, dV = \frac{1}{2}\int_V \varepsilon^T \mathbf{E} \varepsilon \, dV$$

$$= \frac{1}{2}\mathbf{d}^T \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV \mathbf{d}$$

$$= \frac{1}{2}\mathbf{d}^T \mathbf{k} \mathbf{d}$$

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV = tA(\mathbf{B}^T \mathbf{E} \mathbf{B})$$

t: is the thickness of the element.

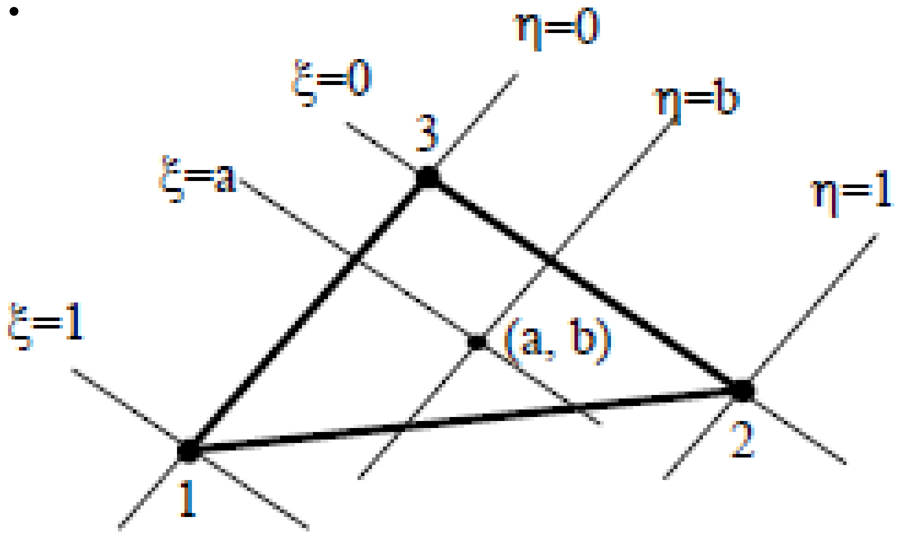
Notice that k for CST is a 6 by 6 symmetric matrix.

The matrix multiplication can be carried out by a computer program.

Natural Coordinates

Natural coordinate system:

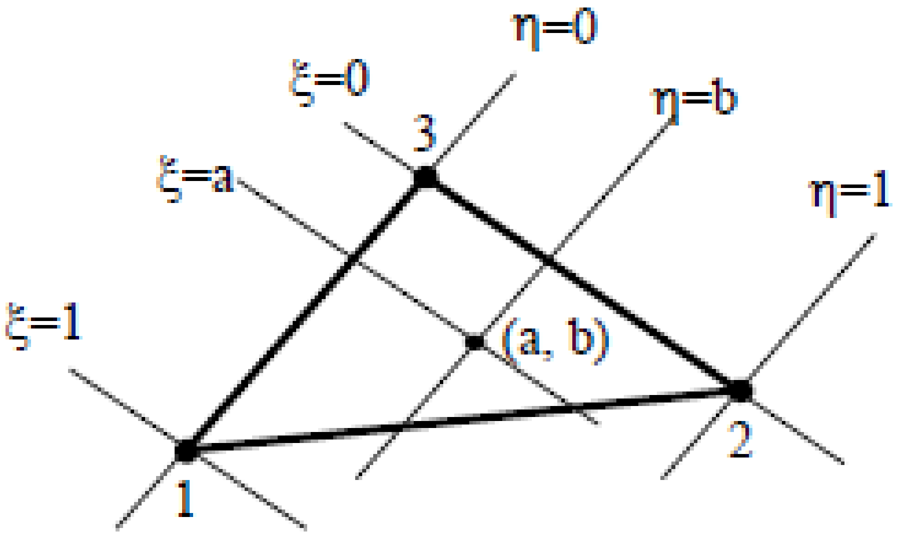
- A local **coordinate** system.
- Specify a point within the element by a set of **dimensionless** numbers whose magnitude **never exceeds unity**.



The Natural Coordinates

Natural Coordinates

Both the expressions of the shape functions and their derivations are lengthy and offer little insight into the behaviour of the element.



The Natural Coordinates

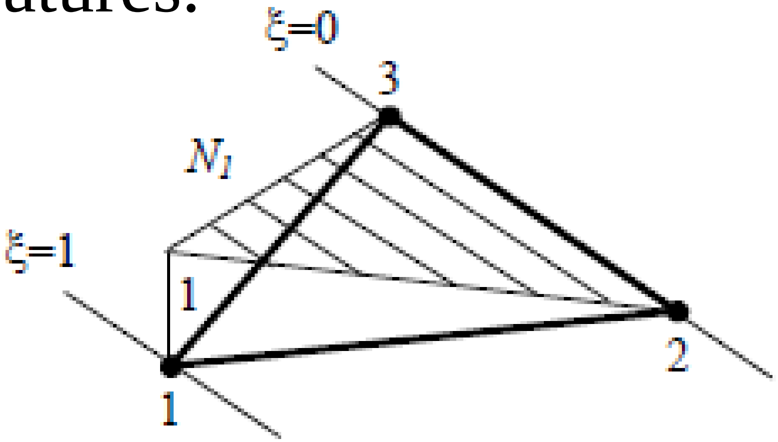
We introduce the natural coordinates (ξ,η) on the triangle, then the shape functions can be represented simply by,

$$N_1 = \xi, \quad N_2 = \eta, \quad N_3 = 1 - \xi - \eta \qquad\qquad N_1 + N_2 + N_3 = 1$$

which ensures that the rigid body translation is represented by the chosen shape functions. Also, as in the 1-D case,

$$N_i = \begin{cases} 1, & \text{at node i;} \\ 0, & \text{at the other nodes} \end{cases}$$

and varies linearly within the element. The plot for shape function $N1$ is shown in the following figure. $N2$ and $N3$ have similar features.



Shape Function N_1 for CST

We have two coordinate systems for the element: the global coordinates (x, y) and the natural coordinates (ξ, η). The relation between the two is given by

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

or,

$$x = x_{13}\xi + x_{23}\eta + x_3$$
$$y = y_{13}\xi + y_{23}\eta + y_3$$

where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$ ($i, j = 1, 2, 3$) as defined earlier.

Displacement u or v on the element can be viewed as functions of (x, y) or (ξ, η). Using the chain rule for derivatives, we have,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix}$$

where **J** is called the *Jacobian matrix* of the transformation.

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}, \qquad \mathbf{J}^{-1} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

where det $\mathbf{J}$ = $x\, y - x\, y = A$ 13 23 23 13 2 has been used

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} u_1 - u_3 \\ u_2 - u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} v_1 - v_3 \\ v_2 - v_3 \end{Bmatrix}$$

Using the results in and the relations
ε = **Du** = **DNd** = **Bd**, we obtain the strain-displacement matrix,

$$\mathbf{B} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

which is the same as we derived earlier